Exercise 7

In Exercises 5–8, show that the given function $u(x)$ is a solution of the corresponding Volterra integral equation:

$$u(x) = 1 - \frac{1}{2}x^2 - \int_0^x (x-t)u(t)\,dt,\ u(x) = 2\cos x - 1$$

Solution

Substitute the function in question on both sides of the integral equation.

$$\begin{aligned}
2\cos x - 1 &\stackrel{?}{=} 1 - \frac{1}{2}x^2 - \int_0^x (x-t)(2\cos t - 1)\,dt \\
&\stackrel{?}{=} 1 - \frac{1}{2}x^2 - \int_0^x (2x\cos t - x - 2t\cos t + t)\,dt \\
&\stackrel{?}{=} 1 - \frac{1}{2}x^2 - \left(\int_0^x 2x\cos t\,dt - \int_0^x x\,dt - \int_0^x 2t\cos t\,dt + \int_0^x t\,dt\right) \\
&\stackrel{?}{=} 1 - \frac{1}{2}x^2 - \left(2x\int_0^x \cos t\,dt - x\int_0^x dt - 2\int_0^x t\cos t\,dt + \left.\frac{t^2}{2}\right|_0^x\right)
\end{aligned}$$

Use integration by parts for the third integral.

$$\begin{aligned}
&\stackrel{?}{=} 1 - \frac{1}{2}x^2 - \left[2x\left.(\sin t)\right|_0^x - x^2 - 2\left.(t\sin t + \cos t)\right|_0^x + \frac{x^2}{2}\right] \\
&\stackrel{?}{=} 1 - \frac{1}{2}x^2 - \left[2x(\sin x - \sin 0) - x^2 - 2(x\sin x + \cos x - \cos 0) + \frac{x^2}{2}\right] \\
&\stackrel{?}{=} 1 - \cancel{\frac{1}{2}x^2} - \left(\cancel{2x\sin x} - \cancel{x^2} - \cancel{2x\sin x} - 2\cos x + 2 + \cancel{\frac{x^2}{2}}\right) \\
&= 2\cos x - 1
\end{aligned}$$

Therefore,

$$u(x) = 2\cos x - 1$$

is a solution of the Volterra integral equation.